



B. Sc. Examination by Course Unit Sample Paper

MAS 117 Introduction to Algebra

Duration: 2 hours

Date and time: ???

The paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

DO NOT TURN OVER THE PAGE UNTIL INSTRUCTED TO DO SO BY THE INVIGILATOR.

Section A: You should attempt all questions. Marks awarded are shown next to the questions. This part of the examination carries 60% of the marks.

Question 1 True or false? Give reasons.

- (a) The square of any integer is congruent to 0 or 1 mod 4.
- (b) For any natural number $n \geq 3$, $(n^2 + n + 2)/2$ is a prime number.

Question 2 Let S be the set of all 2×2 real matrices of the form $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ for real numbers θ .

- (a) Let A and B be two matrices in S . Prove that $AB \in S$.
- (b) Is multiplication of matrices in S commutative?

Question 3 Use Euclid's algorithm to find the greatest common divisor of the real polynomials $f(x) = x^4 + x^3 + x^2 - 1$ and $g(x) = x^3 + 1$.

Question 4 (a) State the Factor Theorem.

- (b) Let $f(x) = x^3 + 2x + 1$ be a polynomial with coefficients in $\mathbb{Z}_3$. Show that $f(x)$ is irreducible in $\mathbb{Z}_3[x]$.

Question 5 (a) Let $(G, \circ)$ be a group. Show that $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$ for all $g, h \in G$.

- (b) Suppose that a group G satisfies $(g \circ h)^2 = g^2 \circ h^2$ for all $g, h \in G$. Show that G is abelian.

Section B: You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best TWO questions answered will be counted. This part of the examination carries 40% of the marks.

Question 6 (a) Define the *complex numbers*, and define the operations of *addition* and *multiplication* on them. Define the *modulus* and *argument* of a complex number.

(b) Prove that

$$(\cos \theta + i \sin \theta)(\cos \phi + i \sin \phi) = \cos(\theta + \phi) + i \sin(\theta + \phi),$$

and deduce *De Moivre's Theorem*

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

(c) Find all complex numbers z satisfying $z^3 = 1$.

Question 7 (a) What is meant by a relation R on a set A ? What does it mean to say that a relation R is (i) reflexive, (ii) symmetric, (iii) transitive?

(b) What is an *equivalence relation*, and what are its *equivalence classes*?

(c) Show that, if R is an equivalence relation on A , then the equivalence classes of R form a partition of A .

(d) Show that the relation R on $\{1, 2, 3, 4, 5\}$ defined by

$$(a, b) \in R \text{ if and only if } a = b \text{ or } a + b = 6$$

is an equivalence relation, and find its equivalence classes.

Question 8 (a) Let p be a prime number. What is meant by the set $\mathbb{Z}_p$ of *integers mod p* ?

(b) Prove that every non-zero element of $\mathbb{Z}_p$ has an inverse in $\mathbb{Z}_p$.

(c) Hence or otherwise prove *Fermat's Little Theorem*: a^p is congruent to $a \pmod{p}$ for every integer a .

(d) Find the inverse of $[10]_{97}$ in $\mathbb{Z}_{97}$.

Question 9 (a) Write down the axioms for a *ring*. What additional conditions are required for a ring R to be

- (i) a commutative ring,
- (ii) a ring with identity,
- (iii) a field?

(b) Let S be a set, and let R denote the set of all subsets of S . Define addition and multiplication on R by the rules

$$A + B = A \triangle B \text{ (symmetric difference),}$$

$$AB = A \cap B \text{ (intersection).}$$

You may assume without proof that R is a ring. Is it commutative? Does it have an identity? Is it a field? Show that every element $A \in R$ satisfies $A^2 = A$.

(c) Now let R be any ring in which $a^2 = a$ holds for every element $a \in R$. Prove that

- (i) R is commutative,
- (ii) $a + a = 0$ for all $a \in R$.

Question 10 (a) What is meant by a *permutation* of the set $\{1, 2, \dots, n\}$? How many permutations of this set are there?

(b) Define the operation of *composition* of permutations, and show that the set of permutations of $\{1, \dots, n\}$ with this operation is a group. (This group is the *symmetric group* S_n .)

(c) Show that S_n is non-abelian if $n \geq 3$.

(d) Give a rule for finding the *order* of an element of S_n , and find all possible orders of elements of S_5 .